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B. Sc. (Honrs) Part 1Paper 1

Subject Mathematics

Title/Heading of topic : Definition and example of partial and total order relation

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Partial orderings

[Def] A relation R on a set S is called *partial ordering* or *partial order* it is is *reflexive*, *antisymmetric*, and *transitive*.

A set S together with a partial ordering R is called *poset* (*partially ordered set*) denotation: (S,R)

Members of S are called *elements of the poset*.

Example: "*greater than or equal*" relation ($\geq$) is a partial ordering on the set of integers. note that R is $\geq$

reflexivity: $a \geq a, \forall a \in \mathbf{Z}$, hence $(a,a) \in R$ or aRa

transitivity: if $a \geq b$ and $b \geq c$, then obviously $a \geq c, \forall a,b,c \in \mathbf{Z}$. Hence if $(a,b),(b,c) \in R$ then $(a,c) \in R$

antisymmetry: if $a \geq b$ and $b \geq a$, then obviously $a = b$

Convention:

In different *posets*, different symbols are used for partial ordering ($\leq$, $\geq$, $\supseteq$, $\subseteq$, $|$).

The notation $\preccurlyeq$ is used to denote that $(a,b) \in R$ in an arbitrary *poset* (S,R)

Note that $\preccurlyeq$ doesn't stand for "less than or equals" relation. It denotes the relation in **any** *poset*.

[Def] The elements a and b of *poset* $(S,\preccurlyeq)$ are called *comparable* is either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise they are called *incomparable*.

Example: In the *poset* $(Z^+, |)$, are the given pairs of integers comparable?

a) *2* and *8*

2 and *8* are comparable because *2* | *8*

b) *21* and *7*

21 and *7* are comparable because *7* | *21*

c) *5* and *13*

5 and 13 are incomparable because neither *5* | *13* nor *13* | *5*

The adjective *partial* is used to describe partial orderings because pairs of elements may be *incomparable*.

When every two elements of the set are comparable, the relation is called a *total ordering*.

[Def] If $(S, \preccurlyeq)$ is a poset and every two elements are comparable, S is called *totally ordered* (*linear ordered*) set, and $\preccurlyeq$ is called a *total order* (*liner order*)*comparable*

A totally ordered set is also called a *chain*.

Example 2: The poset $(S, |)$ is not totally ordered. For example $5 \nmid 13$ and $13 \nmid 5$, i.e. 5 and 13 are incomparable.

[Def] $(S, \preccurlyeq)$ is a *well-ordered set* if it is a *poset* such that $\preccurlyeq$ is a *total ordering* and every non-empty subset of S has at least one element.

Examples:

1) the set of of integers **Z**, with the usual $\leq$ ordering, is *not well-ordered*, because the set of negative integers is a subset of **Z**, but doesn't have the smallest element.

2) The set of positive integers $\mathbf{Z}^+$, with the usual $\leq$ ordering, is *well-ordered*.